Implicit differentiation, second derivatives

Before, when we needed to find the derivative of a function of y in terms of x, like $y = f(x)$, we used our derivative rules to find the first derivative, and then we took the derivative of the first derivative to get the second derivative.

When we have an equation in terms of x and y that can't be easily solved for y, we can use implicit differentiation to find its first and second derivative. Remember that we'll use implicit differentiation to take the first derivative, and then use implicit differentiation again to take the derivative of the first derivative to find the second derivative.

Once we have an equation for the second derivative, we can always make a substitution for y', since we already found y' when we found the first derivative.

Example

Find the second derivative.

$$2y^2 + 6x^2 = 76$$

Because it's a little tedious to isolate y in this equation, we'll use implicit differentiation to take the derivative. Remember that we have to multiple by y' or dy/dx whenever we take the derivative of y.

The first derivative is

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$$4y \cdot y' + 12x = 0$$

$$4yy' = -12x$$

$$y' = \frac{-12x}{4y}$$

$$y' = -\frac{3x}{y}$$

We're looking for the second derivative, y'', of our original function, so we need to take the derivative of y'. Using quotient rule and implicit differentiation together, the second derivative is

$$y'' = -\frac{(3)(y) - (3x)(1 \cdot y')}{(y)^2}$$

$$y'' = -\frac{3y - 3xy'}{y^2}$$

Since the first derivative we found earlier gives us a value for y', we can plug that into the second derivative and then simplify to get our final answer.

$$y'' = -\frac{3y - 3x\left(-\frac{3x}{y}\right)}{y^2}$$

$$y'' = -\frac{3y + \frac{9x^2}{y}}{y^2}$$

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$$y'' = -\frac{\frac{3y^2}{y} + \frac{9x^2}{y}}{y^2}$$

$$y'' = -\frac{\frac{9x^2 + 3y^2}{y}}{y^2}$$

$$y'' = -\frac{9x^2 + 3y^2}{y}\left(\frac{1}{y^2}\right)$$

$$y'' = -\frac{9x^2 + 3y^2}{y^3}$$

Let's try one that's a little more complex.

Example

Find the second derivative.

$$3y^2x + 24x^3 = 6y^4$$

We'll use implicit differentiation, remembering to multiple by y' or dy/dx whenever we take the derivative of y. Remember to use product rule for $3y^2x$.

$$\left(6y \cdot \frac{dy}{dx}\right)(x) + (3y^2)(1) + 72x^2 = 24y^3 \cdot \frac{dy}{dx}$$

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$$6xy\left(\frac{dy}{dx}\right)+3y^2+72x^2=24y^3\left(\frac{dy}{dx}\right)$$

$$6xy\left(\frac{dy}{dx}\right)-24y^3\left(\frac{dy}{dx}\right)=-72x^2-3y^2$$

$$\left(\frac{dy}{dx}\right)(6xy-24y^3)=-72x^2-3y^2$$

$$\frac{dy}{dx}=\frac{-72x^2-3y^2}{6xy-24y^3}$$

$$\frac{dy}{dx}=-\frac{24x^2+y^2}{2xy-8y^3}$$

To find the second derivative, we'll use quotient rule and implicit differentiation together.

$$\frac{d^2y}{dx^2}=-\frac{\left[48x+2y\left(\frac{dy}{dx}\right)\right]\left(2xy-8y^3\right)-\left(24x^2+y^2\right)\left[\left((2)(y)+(2x)\left(1\left(\frac{dy}{dx}\right)\right)\right)-24y^2\left(\frac{dy}{dx}\right)\right]}{\left(2xy-8y^3\right)^2}$$

$$\frac{d^2y}{dx^2}=-\frac{\left[48x+2y\left(\frac{dy}{dx}\right)\right]\left(2xy-8y^3\right)-\left(24x^2+y^2\right)\left[2y+2x\left(\frac{dy}{dx}\right)-24y^2\left(\frac{dy}{dx}\right)\right]}{\left(2xy-8y^3\right)^2}$$

We can use our first derivative to substitute for dy/dx.

$$\frac{d^2y}{dx^2}=-\frac{\left[48x+2y\left(-\frac{24x^2+y^2}{2xy-8y^3}\right)\right]\left(2xy-8y^3\right)-\left(24x^2+y^2\right)\left[2y+2x\left(-\frac{24x^2+y^2}{2xy-8y^3}\right)-24y^2\left(-\frac{24x^2+y^2}{2xy-8y^3}\right)\right]}{\left(2xy-8y^3\right)^2}$$

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$$\frac{d^2y}{dx^2} = -\frac{\left[48x - 2y\left(\frac{24x^2+y^2}{2xy-8y^3}\right)\right](2xy-8y^3) - 2(24x^2+y^2)\left[y - x\left(\frac{24x^2+y^2}{2xy-8y^3}\right) + 12y^2\left(\frac{24x^2+y^2}{2xy-8y^3}\right)\right]}{(2xy-8y^3)^2}$$

This is a huge answer, but that's not unusual when you're dealing with complex implicit differentiation. So just try to simplify as much as you can, but don't worry too much about getting a really clean answer.

$$\frac{d^2y}{dx^2} = -\frac{\left(48x - \frac{48x^2y+2y^3}{2xy-8y^3}\right)(2xy-8y^3) - (48x^2+2y^2)\left(y - \frac{24x^3+xy^2}{2xy-8y^3} + \frac{288x^2y^2+12y^4}{2xy-8y^3}\right)}{(2xy-8y^3)^2}$$

$$\frac{d^2y}{dx^2} = -\frac{\left(48x - \frac{48x^2y+2y^3}{2xy-8y^3}\right)(2xy-8y^3) - (48x^2+2y^2)\left(y - \frac{24x^3+xy^2-288x^2y^2-12y^4}{2xy-8y^3}\right)}{(2xy-8y^3)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(48x^2+2y^2)\left(y - \frac{24x^3+xy^2-288x^2y^2-12y^4}{2xy-8y^3}\right) - \left(48x - \frac{48x^2y+2y^3}{2xy-8y^3}\right)(2xy-8y^3)}{(2xy-8y^3)^2}$$

$$\frac{d^2y}{dx^2} = \frac{48x^2y - \frac{48x^2(24x^3+xy^2-288x^2y^2-12y^4)}{2xy-8y^3} + 2y^3 - \frac{2y^2(24x^3+xy^2-288x^2y^2-12y^4)}{2xy-8y^3} - \left[96x^2y - 384xy^3 - \frac{2xy(48x^2y+2y^3)}{2xy-8y^3} + \frac{8y^3(48x^2y+2y^3)}{2xy-8y^3}\right]}{(2xy-8y^3)^2}$$

$$\frac{d^2y}{dx^2} = \frac{48x^2y - \frac{48x^2(24x^3+xy^2-288x^2y^2-12y^4)}{2xy-8y^3} + 2y^3 - \frac{2y^2(24x^3+xy^2-288x^2y^2-12y^4)}{2xy-8y^3} - 96x^2y + 384xy^3 + \frac{2xy(48x^2y+2y^3)}{2xy-8y^3} - \frac{8y^3(48x^2y+2y^3)}{2xy-8y^3}}{(2xy-8y^3)^2}$$

$$\frac{d^2y}{dx^2} = \frac{2y^3 - 48x^2y + 384xy^3 + \frac{2xy(48x^2y+2y^3) - 8y^3(48x^2y+2y^3) - 48x^2(24x^3+xy^2-288x^2y^2-12y^4) - 2y^2(24x^3+xy^2-288x^2y^2-12y^4)}{2xy-8y^3}}{(2xy-8y^3)^2}$$

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$$\frac{d^2y}{dx^2}=\frac{2y^3-48x^2y+384xy^3+\dfrac{(2xy-8y^3)(48x^2y+2y^3)-(48x^2+2y^2)(24x^3+xy^2-288x^2y^2-12y^4)}{2xy-8y^3}}{(2xy-8y^3)^2}$$

$$\frac{d^2y}{dx^2}=\frac{\dfrac{(2y^3-48x^2y+384xy^3)(2xy-8y^3)}{2xy-8y^3}+\dfrac{(2xy-8y^3)(48x^2y+2y^3)-(48x^2+2y^2)(24x^3+xy^2-288x^2y^2-12y^4)}{2xy-8y^3}}{(2xy-8y^3)^2}$$

$$\frac{d^2y}{dx^2}=\frac{\dfrac{(2y^3-48x^2y+384xy^3)(2xy-8y^3)+(2xy-8y^3)(48x^2y+2y^3)-(48x^2+2y^2)(24x^3+xy^2-288x^2y^2-12y^4)}{2xy-8y^3}}{(2xy-8y^3)^2}$$

$$\frac{d^2y}{dx^2}=\frac{(2y^3-48x^2y+384xy^3)(2xy-8y^3)+(2xy-8y^3)(48x^2y+2y^3)-(48x^2+2y^2)(24x^3+xy^2-288x^2y^2-12y^4)}{(2xy-8y^3)^3}$$

$$\frac{d^2y}{dx^2}=\frac{(2xy-8y^3)\left[(2y^3-48x^2y+384xy^3)+(48x^2y+2y^3)\right]-(48x^2+2y^2)(24x^3+xy^2-288x^2y^2-12y^4)}{(2xy-8y^3)^3}$$

$$\frac{d^2y}{dx^2}=\frac{(2xy-8y^3)(4y^3+384xy^3)-(48x^2+2y^2)(24x^3+xy^2-288x^2y^2-12y^4)}{(2xy-8y^3)^3}$$

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